

# MOUNTAIN PASS THEOREM WITH INFINITE DISCRETE SYMMETRY

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**ABSTRACT.** We extend an equivariant Mountain Pass Theorem, due to Bartsch, Clapp and Puppe for compact Lie groups to the setting of infinite discrete groups satisfying a maximality condition on their finite subgroups. As an application, we examine the critical set of generalizations of periodic functionals first studied by Rabinowitz, [30], as well as the solutions to symmetric systems of ODE associated with them.

Since the early days of Variational Analysis, symmetries have played a fundamental role in the analysis of critical points and sets of functionals [1], [17], [9]. The development of Equivariant Algebraic Topology, particularly Equivariant Homotopy Theory, has given a number of tools to conclude the existence of critical points in problems which are invariant under the action of a compact Lie group.

The extension of methods of Equivariant Topology to the setting of actions of infinite groups and their applications to problems in nonlinear analysis is the object of this paper. The main result of this note is the modification of a result by Bartsch and Clapp originally proven for compact Lie groups, to infinite discrete groups with appropriate families of finite subgroups inside them. Our methods differ from that of [9] in that we use classifying spaces for families, the analysis of the behavior of elements under the differentials in the Atiyah-Hirzebruch spectral sequence, a notion of “universal proper length”, as well as the use of an equivariant quantitative deformation Lemma, which replaces the typical apriori compactness assumptions related to either the Palais-Smale condition or the  $G$ -strong deformation property of [9]. See example 2.1 for more on symmetric mountain pass situations without apriori compactness hypothesis.

**Theorem -1.1** (Mountain Pass Theorem). *Let  $G$  be an infinite discrete group acting linearly on a real Banach space  $E$  of infinite dimension. Suppose that  $G$  satisfies the maximality condition -1.2 and that the linear action is proper outside 0. Let  $\phi : E \rightarrow \mathbb{R}$  be a  $G$ -invariant functional. For any value  $a \in \mathbb{R}$ , define the set below  $a$ ,  $\phi^a = \{x \in E \mid \phi(x) \leq a\}$  and the critical set  $K = \bigcup_{c \in \mathbb{R}} K_c$ , where  $K_c$  is the critical set at level  $c$ ,  $K_c = \{u \mid \|\phi'(u)\| = 0, \phi(u) = c\}$ . Suppose that*

- $\phi(0) \leq a$  and there exists a linear subspace  $\hat{E} \subset E$  of finite codimension such that  $\hat{E} \cap \phi^a$  is the disjoint union of two closed subspaces one of which is bounded and contains 0.
- The functional  $\phi$  satisfies a quantitative deformation property with respect to the subset  $S \subset E$ , which is assumed to be a fundamental deformation region.
- The group  $G$  satisfies the maximal finite subgroups condition -1.2.

*Then, the equivariant Lusternik-Schnirelmann category of  $E$  relative to  $\phi^a$ ,  $G - \text{cat}(\phi^a, E)$  is infinite. If moreover, the critical sets of isolated critical values  $K_c$*

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are cocompact under the group action, meaning that the quotient spaces  $G \backslash K_c$  are compact, then  $\phi(K)$  is unbounded above.

Conditions -1.2 restrict the maximal finite subgroups and their conjugacy relations.

**Condition -1.2.** Let  $\mathcal{M}$  be the subset of maximal finite subgroups.

- There exists a prime number  $p$  such that every nontrivial finite subgroup is contained in a unique maximal  $p$ - group  $M \in \mathcal{MAX}$ .
- $M \in \mathcal{MAX} \implies N_G(M) = M$

Notice that in particular, the finite subgroups of  $G$  are all finite  $p$ - groups.

These conditions are satisfied in several important cases. Among them:

- (i) Extensions  $1 \rightarrow \mathbb{Z}^n \rightarrow G \rightarrow K \rightarrow 1$  by a finite  $p$ - group given by a representation  $K \rightarrow \mathbb{Z}^n$  acting free outside from the origin [25], Lemma 6.3.
- (ii) Fuchsian groups, more generally NEC (non-euclidean crystallographic groups ) for which the isotropy consists only of  $p$ -groups. [25].
- (iii) One relator groups  $G = \langle q_i \mid r \rangle$  for which the finite subgroups consists of  $p$ -groups. See [26], Propositions 5.17, 5.18, 5.19. in pages 107 and 108.

The quantitative deformation property with respect to a set  $S$  0.6 is an equivariant Version of Willem's deformation lemma, 2.3 in [34], page 38. We use a consequence of it, that if the set  $S$  is a fundamental deformation region, (see definition 0.7), then there exists deformations of sets above isolated critical values out of neighborhoods of the critical set, see Corollary 0.8.

This paper is organized as follows: in the first section, the usual facts concerning the relation between critical points, cohomology length, Lusternik-Schnirelmann category and equivariant deformation theorems are stated, being modified slightly from [34] lemma 2.3 page 38, respectively [4], [12]. We introduce the notion of Universal Proper Length.

In the second section, we use some algebraic properties of the classifying space for proper actions of groups with an appropriate family of maximal finite subgroups in order to conclude the unboundedness of critical values.

This is done adapting a construction of elements in the Burnside Ring of a finite group, originally due to Bartsch, Clapp and Puppe [9] to the infinite group setting, using mainly the Atiyah-Hirzebruch spectral sequence, as well as a version of the Segal Conjecture for families of finite groups inside infinite groups [20], [6].

Finally, we present an example of a Banach space with an action of infinite discrete groups to illustrate the applications of the methods handled in this work, extending results of Rabinowitz [30].

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## 0. PROPER LUSTERNIK-SCHNIRELMANN CATEGORY, UNIVERSAL PROPER LENGTH AND CRITICAL POINTS

The notion of a proper  $G$ -space provides the adequate setting for the study of non compact transformation groups.

**Definition 0.1.** Let  $G$  be a second countable, Hausdorff locally compact group. Let  $X$  be a second countable, locally compact Hausdorff space. Recall that a  $G$ -space is proper if the map

$$\begin{aligned} G \times X &\rightarrow X \times X \\ (g, x) &\mapsto (x, gx) \end{aligned}$$

is proper.

**Definition 0.2.** Recall that a  $G$ -CW complex structure on the pair  $(X, A)$  consists of a filtration of the  $G$ -space  $X = \cup_{-1 \leq n} X_n$ ,  $X_{-1} = \emptyset, X_0 = A$  and for which every space  $X_n$  is inductively obtained from the previous one by attaching cells in pushout diagrams of the form

$$\begin{array}{ccc} \coprod_i S^{n-1} \times G/H_i & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_i D^n \times G/H_i & \longrightarrow & X_n \end{array}$$

We say that a proper  $G$ -CW complex is finite if it consists of a finite number of cells  $G/H \times D^n$ .

The following result enumerates some facts which will be needed in the following, which are proven in chapter one of [21]:

**Proposition 0.3.** *Let  $(X, A)$  be a proper  $G$ -CW pair*

- (i) *The inclusion  $A \rightarrow X$  is a closed cofibration.*
- (ii)  *$A$  is a neighborhood  $G$ -deformation retract, in the sense that there exists a neighborhood  $A \subset U$ , of which  $A$  is a  $G$ -equivariant deformation retract. The neighborhood can be chosen to be closed or open. In particular, all  $G$ -CW complexes are  $G$ -ANR.*

**Remark 0.4.** In the case of Lie groups, a proper action amounts to the fact that all isotropy subgroups are compact and that a local triviality condition, coded in the Slice Theorem is satisfied [28]. Specializing to Lie groups acting properly on  $G$ -CW complexes, the conditions boil down to the fact that all stabilizers are compact, see [21], Theorem 1.23. In particular for a cellular action of a discrete group  $G$  on a  $G$ -CW complex, a proper action reduces to the finiteness of all stabilizer groups. Notice that any (continuous) action of a compact Lie group or a finite group on a locally compact, Hausdorff space is proper.

The notion of  $G$ -category of a space with an action of a compact Lie group was introduced by Marzantowicz in [27]. Ayala-Lasheras-Quintero [4] introduced the corresponding notion for proper actions of Lie Groups. From this point on, all  $G$ -spaces considered are  $G$ -Absolute Neighborhood Retracts ( $G$ -ANR).  $G$ -CW complexes are examples of  $G$ -ANR. A proper  $G$ -ANR is called cocompact if the quotient space  $G \backslash X$  is compact.

**Definition 0.5.** Let  $X$  be a Proper  $G$ -space. A  $G$ -invariant subset  $A \subset X$  is said to be of  $G$ -category 1 if there exists an orbit  $Gx$  such that the inclusion  $A \rightarrow X$  is  $G$ -homotopic to a map  $A \rightarrow Gx \rightarrow X$ . The  $G$ -category of  $A$  in  $X$  is the smallest number  $k$  such that there exists a cover  $A_j$  of  $A$  consisting of closed  $G$ -subsets of  $G$ -category 1. More generally, the  $G$ -category of an equivariant map  $f : X \rightarrow Y$  between proper  $G$ -spaces is the smallest  $k$  such that there exists a cover  $\{X_1, \dots, X_k\}$  consisting of closed  $G$ -sets and  $G$ -orbits  $Gx_i$  such that the restrictions  $f|_{X_i}$  are  $G$ -homotopic in  $Y$  to a map  $X_i \rightarrow Gx_i \rightarrow Y$ .

In the case of a  $G$ -ANR, the following definition of category is more handable, and equivalent to the previous one, given in terms of closed sets.

The  $G$ -equivariant category of a map  $f : (X, X') \rightarrow (Y, Y')$ , denoted  $G\text{-cat}(f)$  is the smallest number  $k$  such that  $X$  can be covered by  $k + 1$  open subspaces  $X_0, X_1, \dots, X_k$  with the following properties:

- $X' \subset X_0$  and there is a homotopy  $h : (X_0, X') \times I \rightarrow (Y, Y')$  with  $h|_{0} = f|_{X_0}$  and  $h|_{1} \in Y'$  for all  $x \in X_0$ .

- For every  $i \in \{1, \dots, k\}$  there exist  $G$ -maps  $\alpha_i : X_i \rightarrow A_i$  and  $\beta_i : A_i \rightarrow Y$  with  $A_i$  a  $G$ -orbit  $G/H_i$  such that the restriction of  $f$  to  $X_i$  is the is  $G$ -homotopic to the composition  $\beta_i \circ \alpha_i$

If no such a number exists, then we write  $G - cat(f) = \infty$ .

The next ingredient for a Mountain Pass Theorem Type result is a deformation lemma. This is achieved using the notion of pseudogradient vector field, and the quantitative deformation lemma of Willem, lemma 2.3 in [34], page 37, interpreted in an equivariant setting. In the following we will deal with a  $G$ -Banach Space  $X$  with a proper action outside 0 of the discrete group  $G$ . Recall [28], the existence of  $G$ -invariant metrics  $d$ . If  $S \subset X$  is a set, the  $\delta$ -inflated set  $S_\delta$  is defined to be the set  $\{y \in X \mid d(y, S) \leq \delta\}$ .

**Lemma 0.6** (Equivariant Quantitative Deformation). *Let  $G$  be a discrete group acting properly outside of the origin on a Banach space  $X$ . Let  $\Phi : X \rightarrow \mathbb{R}$  a  $G$ -invariant  $C^1$ -functional,  $S \subset X$ ,  $c \in \mathbb{R}$ ,  $\epsilon > 0$  such that for any  $u \in \Phi^{-1}[c - 2\epsilon, c + 2\epsilon] \cap S_{2\delta}$  the inequality*

$$|\Phi'(u)| \geq \frac{8\epsilon}{\delta}$$

*holds. Then, there exists a  $G$ -equivariant deformation  $\eta : X \times I \rightarrow X$  satisfying:*

- (i) *For all  $t$ , the map  $\eta_t := \eta(\cdot, t) : X \rightarrow X$  is an homeomorphism.*
- (ii)  *$\eta(u, 0) = u$*
- (iii)  *$\eta(t, u) = u$  whenever  $u \notin \Phi^{-1}[c - 2\epsilon, c + 2\epsilon] \cap S_{2\delta}$ .*
- (iv)  *$\eta(\phi^{c+\epsilon} \cap S, 1) \subset \Phi^{c-\epsilon}$ .*
- (v)  *$|\eta(t, u) - u| < \delta$ .*
- (vi)  *$\phi(\eta(t, u)) < c$   $u \in \phi^c \cap S_\delta$ .*

*Proof.* Let  $W$  be a locally Lipschitz pseudogradient vector field associated to the functional  $\Phi$ , which is known to exist, se [29], Lemma A.2 in Appendix A, page 80. Put  $W_G(v) = \sum_{g \in G} g^{-1}(W(gv))$ . Notice that due to the slice theorem, this is a locally finite sum, namely indexed by the elements of a finite stabilizer of a point outside from 0. Since the metric might be assumed to be invariant under the  $G$ -action (yet another consequence of properness),  $W_G$  is a  $G$ -equivariant locally Lipschitz pseudogradient vector field on  $\tilde{X} = W - \{v \mid \Phi'(v) = 0\}$

The proper action allows to construct a locally Lipschitz function  $\varphi : X \rightarrow [0, 1] \subset \mathbb{R}$  such that  $\varphi|_{\Phi^{-1}([c-2\epsilon, c+2\epsilon] \cap S_\delta)} \equiv 1$  and  $\varphi|_{S_{2\delta} \cup X - \Phi^{-1}([c-2\epsilon, c+2\epsilon])} \equiv 0$ . Define the  $G$ -equivariant maximal descent vector field  $\psi : X \rightarrow X$

$$\psi(v) = \begin{cases} \frac{-\varphi(v)}{|W_G(v)|} W_G(v) & v \in \Phi^{-1}([c - 2\epsilon, c + 2\epsilon]) \cap S_{2\delta} \\ 0 & \text{otherwise} \end{cases}$$

The  $G$ -equivariant problem  $\dot{w}(t) = \psi(w(t))$  with the initial condition  $w(0) = u$  admits a solution defined on  $[0, \infty)$ . It is then routine that the deformation defined by  $\eta(t, u) = w(\delta t, u)$  satisfies i to iv. □

In this situation, we will say that the functional has the quantitative deformation property relative to the subspace  $S$  at the level  $c$ .

**Definition 0.7.** Let  $\phi$  be a functional with a quantitative deformation property with respect to the Set  $S$ . A Set  $S$  is called a fundamental deformation region if:

- For any  $c$ , the set  $\hat{K}_c := K_c \cap S$  is compact.
- For any  $c$ ,  $\hat{K}_c$  is a fundamental region for the critical set at level  $c$ ,  $K_c$ , i.e., the action of  $G$  on  $\hat{K}_c$  gives all of  $K_c$

If the set  $S$  happens to be such that  $K_c \cap S$  is a fundamental deformation region, then the following refinement of the quantitative deformation is possible:

**Corollary 0.8.** *Let  $\phi : X \rightarrow \mathbb{R}$  be a functional having an quantitative deformation property relative to the fundamental deformation region  $S$ . Then, for any invariant neighborhood  $U$  of the critical set  $K_c$ , there exist  $\epsilon > 0$  and a continuous equivariant map  $\eta : I \times X, X$  such that*

- $\eta(u, 0) = u$ .
- $\eta(1, \phi^{c+\epsilon} - U) \subset \phi^{c-\epsilon}$ .
- If  $K_c = \emptyset$ , then  $\eta(1, \phi^{c+\epsilon}) \subset \phi^{c-\epsilon}$ .

*Proof.* Since  $K_c \cap S$  is compact, there exists  $\delta > 0$  such that  $(K_c \cap S)_\delta \subset U$ . The  $G$ -invariance of  $U$  and the fact that  $S$  is a fundamental deforming region imply that  $K_{c_\delta} = G(K_c \cap S)_\delta \subset U$ . The compactness of  $K_c \cap S$  implies that there exist a number  $b > 0$  and  $\epsilon$  such that  $|\phi'(q)| > b$  for  $q \in \phi^{-1}[c - \epsilon, c + \epsilon] \cap S_\delta$ . Choose  $0 < \epsilon < 1$  such that  $0 < \frac{\delta\epsilon}{b} < b$ , then the deformation obtained from the deformation lemma satisfies the required conditions.  $\square$

**Proposition 0.9.** *Let  $M$  be a proper, paracompact  $G$ -Banach,  $C^1$ -manifold. Let  $\phi : M \rightarrow \mathbb{R}$  be a  $G$ -invariant  $C^1$ -function satisfying the deformation property with respect to neighborhoods of critical sets associated to isolated critical values. Suppose that, for every isolated critical value  $c$ , the critical set  $K_c$  is cocompact.*

- *If the function is bounded below, then the number of critical points of  $\phi$  with values  $> a$  in  $M$  is at least  $G - \text{cat}(\phi^a, M)$ .*
- *If  $G - \text{cat}(M)$  is greater than the number of critical values of  $\phi$  above  $a$ , then there is at least one  $c > a$  such that the critical set  $K_c$  has positive covering dimension, in particular  $\phi$  has infinitely many critical orbits with values above  $a$ .*
- *If  $G - \text{cat}(K, M) = \infty$ , then  $\phi$  has an unbounded sequence of critical values.*

*Proof.* The proofs given in [12], theorem 2.3 and corollary 2.4, pages 606 and 607, and [13], Theorem 1.1 extend to the proper setting. The point is that the equivariant Lusternik-Schnirelmann Category for proper spaces satisfies subadditivity, deformation monotonicity, and continuity (Proposition 2.3 in [4] in the absolute case, and the obvious modification extends to the relative category).  $\square$

We recall the notion of the classifying space for a family of subgroups.

**Definition 0.10.** Let  $\mathcal{F}$  be a collection of subgroups in a discrete group  $G$  closed under conjugation and intersection. A model for the Classifying Space for the family  $\mathcal{F}$  is a  $G$ -CW complex whose isotropy subgroups all lie in  $\mathcal{F}$  and for which for every subgroup  $H \in \mathcal{F}$ , the fixed point set  $X^H$  is either contractible if  $H \in \mathcal{F}$  or empty otherwise.

Particularly relevant is the classifying space for proper actions, the classifying space for the family  $\mathcal{FIN}$  of finite subgroups, denoted by  $\underline{EG}$ .

The classifying space for proper actions always exists, is unique up to  $G$ -homotopy and admits several models [24].

The following list includes some examples. We remit to [24] for further discussion.

- If  $G$  is a compact group, then the singleton space is a model for  $\underline{EG}$ .
- Let  $G$  be a group acting properly and cocompactly on a CAT(0) space  $X$ , in the sense of [11]. Then  $X$  is a model for  $\underline{EG}$ .
- Let  $G$  be a Coxeter Group. The Davis Complex is a model for  $\underline{EG}$ .
- Let  $G$  be a Mapping Class Group of a surface. The Teichmüller Space is a model for  $\underline{EG}$ .

- Let  $G$  be a Gromov Hyperbolic Group. The Rips Complex is a model for the classifying space for  $\underline{EG}$ .

**Remark 0.11.** The classifying space for proper actions,  $\underline{EG}$  can be characterized by the universal property of being a proper  $G$ -CW-complex  $\underline{EG}$  with the property that for any proper  $G$ -CW complex  $X$ , there exists up to  $G$ -homotopy a unique  $G$ -equivariant map

$$X \rightarrow \underline{EG}$$

Recall the notion of an Equivariant Cohomology Theory, [22].

**Definition 0.12.** Let  $G$  be a group and fix an associative ring with unit  $R$ . A  $G$ -Cohomology Theory with values in  $R$ -modules is a collection of contravariant functors  $\mathcal{H}_G^n$  indexed by the integer numbers  $\mathbb{Z}$  from the category of  $G$ -CW pairs together with natural transformations  $\partial_G^n : \mathcal{H}_G^n(A) := \mathcal{H}_G^n(A, \phi) \rightarrow \mathcal{H}_G^{n+1}(X, A)$ , such that the following axioms are satisfied:

- (i) If  $f_0$  and  $f_1$  are  $G$ -homotopic maps  $(X, A) \rightarrow (Y, B)$  of  $G$ -CW pairs, then  $\mathcal{H}_G^n(f_0) = \mathcal{H}_G^n(f_1)$  for all  $n$ .
- (ii) Given a pair  $(X, A)$  of  $G$ -CW complexes, there is a long exact sequence

$$\begin{aligned} \dots \xrightarrow{\mathcal{H}_G^{n-1}(i)} \mathcal{H}_G^{n-1}(A) \xrightarrow{\partial_G^{n-1}} \mathcal{H}_G^n(X, A) \xrightarrow{\mathcal{H}_G^n(j)} \mathcal{H}_G^n(X) \\ \xrightarrow{\mathcal{H}_G^n(i)} \mathcal{H}_G^n(A) \xrightarrow{\partial_G^n} \mathcal{H}_G^{n+1}(X, A) \xrightarrow{\mathcal{H}_G^{n+1}(j)} \dots \end{aligned}$$

where  $i : A \rightarrow X$  and  $j : X \rightarrow (X, A)$  are the inclusions.

- (iii) Let  $(X, A)$  be a  $G$ -CW pair and  $f : A \rightarrow B$  be a cellular map. The canonical map  $(F, f) : (X, A) \rightarrow (X \cup_f B, B)$  induces an isomorphism

$$\mathcal{H}_G^n(X \cup_f B, B) \xrightarrow{\cong} \mathcal{H}_G^n(X, A)$$

- (iv) Let  $\{X_i \mid i \in \mathcal{I}\}$  be a family of  $G$ -CW-complexes and denote by  $j_i : X_i \rightarrow \coprod_{i \in \mathcal{I}} X_i$  the inclusion map. Then the map

$$\coprod_{i \in \mathcal{I}} \mathcal{H}_G^n(j_i) : \mathcal{H}_G^n(\coprod_i X_i) \xrightarrow{\cong} \coprod_{i \in \mathcal{I}} \mathcal{H}_G^n(X_i)$$

is bijective for each  $n \in \mathbb{Z}$ .

A  $G$ -Cohomology Theory is said to have a multiplicative structure if there exist natural, graded commutative  $\cup$ - products

$$\mathcal{H}_G^n(X, A) \otimes \mathcal{H}_G^m(X, A) \rightarrow \mathcal{H}_G^{n+m}(X, A)$$

Let  $\alpha : H \rightarrow G$  be a group homomorphism and  $X$  be a  $H$ -CW complex. The induced space  $\text{ind}_\alpha X$ , is defined to be the  $G$ -CW complex defined as the quotient space  $G \times X$  by the right  $H$ -action given by  $(g, x) \cdot h = (g\alpha(h), h^{-1}x)$ .

An Equivariant Cohomology Theory consists of a family of  $G$ -Cohomology Theories  $\mathcal{H}_G^*$  together with an induction structure

$$\mathcal{H}_G^n(\text{ind}_\alpha(X, A)) \cong \mathcal{H}_H^n(X, A)$$

for group homomorphisms  $\alpha : H \rightarrow G$  whose kernel acts freely on  $X$  satisfying the following conditions:

- (i) For any  $n$ ,  $\partial_G^n \circ \text{ind}_\alpha = \text{ind}_\alpha \circ \partial_H^n$ .

- (ii) For any group homomorphism  $\beta : G \rightarrow K$  such that  $\ker \beta \circ \alpha$  acts freely on  $X$ , one has

$$\text{ind}_{\alpha \circ \beta} = \mathcal{H}_K^n(f_1 \circ \text{ind}_\beta \circ \text{ind}_\alpha) : \mathcal{H}_K^n(\text{ind}_{\beta \circ \alpha}(X, A)) \rightarrow \mathcal{H}_H^n(X, A)$$

where  $f_1 : \text{ind}_\beta \text{ind}_\alpha \rightarrow \text{ind}_{\beta \circ \alpha}$  is the canonical  $G$ -homeomorphism.

- (iii) For any  $n \in \mathbb{Z}$ , any  $g \in G$ , the homomorphism

$$\text{ind}_{c(g):G \rightarrow G} : \mathcal{H}_G^n(\text{ind}_{c(g):G \rightarrow G}(X, A)) \rightarrow \mathcal{H}_G^n(X, A)$$

agrees with the map  $\mathcal{H}_G^n(f_2)$ , where  $f_2 : (X, A) \rightarrow \text{ind}_{c(g):G \rightarrow G}$  sends  $x$  to  $(1, g^{-1}x)$  and  $c(g)$  is the conjugation isomorphism in  $G$ .

Recall [14], [22], that for any Equivariant Cohomology Theory  $\mathcal{H}^*$  on finite  $G$ -CW complexes there exists a spectral sequence with  $E^2$ -term given in terms of Bredon Cohomology

$$E_2^{p,q} = H_{\mathbb{Z}Or(G)}^p(X, \mathcal{H}_G^{-q}(G/H))$$

converging to  $H_G^*(X)$ .

The existence of equivariant Cohomological Chern characters (or equivalently, the rational collapse of the Atiyah-Hirzebruch spectral sequence) imply:

**Proposition 0.13.** *Let  $X$  be a finite  $G$ -CW complex. For any element*

$$x \in H_{\mathbb{Z}Or(G)}^*(X, \mathcal{H}^j(G/?))$$

*there exists some positive integer  $k$  such that  $x^k$  is contained in the image of  $\mathcal{H}_G^n(X)$  under the edge homomorphism*

$$\text{Edge}_G : \mathcal{H}_G^n(X) \longrightarrow \prod_{i+j=n} H_{\mathbb{Z}Or(G)}^i(X, \mathcal{H}^j(G/?))$$

**Definition 0.14.** (Universal Cohomology Length relative to a family of subgroups)

Let  $\mathcal{A} = \{G/H_i\}$  be a collection of orbit spaces representing all homogeneous  $G$ -spaces with isotropy in some family  $\mathcal{F}$  of subgroups of  $G$ . Let  $M$  be a module over the ring  $\pi_G^0(E_G(\mathcal{F}))$ . The  $\mathcal{H}_{\mathcal{A}}$ -length of the module  $M$  is the smallest number  $k$  such that there exists spaces  $A_1, \dots, A_k \in \mathcal{A}$  such that for any  $\gamma \in M$  and  $\omega_i$  in the kernel of the map

$$\mathcal{H}_G^0(E_G(\mathcal{F})) \rightarrow \mathcal{H}_G^0(G/H_i)$$

given by the composition of the edge homomorphism

$$\mathcal{H}_G^0(E_G(\mathcal{F})) \rightarrow \lim_{H_i} \mathcal{H}_G^0(G/H_i)$$

and the structure map  $\lim_{H_i} \mathcal{H}_G^0(G/H_i) \rightarrow \mathcal{H}_G^0(G/H_i)$ , the product  $\gamma \omega_1 \dots \omega_k$  is zero.

## 1. COMPUTATIONS IN BURNSIDE RINGS

We specialize now to equivariant stable cohomotopy for proper actions.

We give a quick summary of important facts of Equivariant Stable Cohomotopy for finite groups.

**Theorem 1.1.** *Let  $G$  be a finite group. Then*

- *The 0-th equivariant cohomotopy group of a point,  $\pi_G^0(\{\bullet\})$  is isomorphic to the Burnside ring, denoted by  $A(G)$ , the Grothendieck ring of isomorphism classes of finite  $G$ -sets.*

- The Burnside ring  $A(G)$  is provided with maps  $\varphi_H : A(G) \rightarrow \mathbb{Z}$ , each one for every conjugacy class of subgroups in  $G$ . These extend to an injective map  $A(G) \rightarrow \prod_{H \in \text{ccs}(G)} \mathbb{Z}$ , where  $\text{ccs}(G)$  denotes the set of conjugacy classes of subgroups in  $G$ .
- The prime ideals in  $A(G)$  are given by the sets  $\mathcal{P}_{K,p} = \{x \mid \varphi_H(x) \cong 0(p)\}$ ,  $\mathcal{P}_{H,0} = \{x \mid \varphi_H(x) = 0\}$ , where  $p$  is a prime number. The augmentation ideal  $I_G$  is defined as the Ideal  $\{x \mid \varphi_e(x) = 0\}$ .
- There exists an element, the Bartsch element  $0 \neq x \in A(G)$  with the property that  $\varphi_H(x) = 0$  for every subgroup  $H$ .
- If  $p$  is a prime number and  $G$  is a finite  $p$ -group, then the completion map  $A(G) \rightarrow A(G)_{\hat{I}_G}$  is injective and the  $I_G$ -adical topology and the  $p$ -adical topologies coincide.

*Proof.*

- This is well known. See [32], [33].
- See [33], chapter II, section 8, pages 155-160. The image is characterized by a set of congruences for the number of generators of cyclic subgroups of the Weyl groups  $NH/H$  for every conjugacy class of subgroups. [33], section 5 chapter IV, page 256. Alternatively, Theorem 1.3 in [18], page 41.
- This is proven in [18], page 43, [15].
- This is done in [9]. The element is constructed as follows: let  $K$  be a proper subgroup of  $G$ . Put  $u_K = [G/K] - |G/K|^K [G/G]$ . The element  $x$  is defined as the product of all such  $u_K$ , each one for every conjugacy class of subgroups in  $K$ .
- For a detailed proof see [18]. The first result, Corollary 1.11 in [18], follows from the fact that in this situation the kernel of the completion map,  $\cap_n I_G^n$  coincides with  $\cap \ker(\varphi_U)$ , where  $U$  ranges among all  $p$ -syllow groups. The second result follows from Frobenius reciprocity and an analysis of the congruences defining the Burnside ring as subring inside  $\prod_{H \in \text{ccs}(G)} \mathbb{Z}$ , proposition 1.12 in [18], page 44.

□

Equivariant Cohomotopy for proper actions of infinite discrete groups on finite  $G$ -CW complexes was defined in [23] via finite dimensional equivariant vector bundles for proper, finite  $G$ -CW complexes. Alternative approaches are given by the nonlinear cocycles of [5], which allow actions of noncompact Lie groups on finite  $G$ -CW complexes, or the spectrum version of [7], which allow actions of discrete groups on proper and possibly infinite  $G$ -CW complexes. All three of these approaches are compared in [6]. For convenience, we give the definition from [23]:

**Definition 1.2.** A  $G$ -vector bundle over a  $G$ -CW-complex  $X$  consists of a real vector bundle  $\xi : E \rightarrow X$  together with a  $G$ -action on  $E$  such that  $\xi$  is equivariant and each  $g \in G$  acts on  $E$  and  $X$  via vector bundle isomorphisms.

Let  $S^\xi$  denote its fibrewise one-point compactification.

**Definition 1.3.** Let  $X$  be a proper  $G$ -CW-complex. Let  $\text{SPHB}^G(X)$  be the category with

- $\text{Ob}(\text{SPHB}^G(X)) = \{G\text{-vector bundles over } X\}$ ; and
- a morphism from a vector bundle  $\xi : E \rightarrow X$  to vector bundle  $\mu : F \rightarrow X$  is given by a bundle map  $u : S^\xi \rightarrow S^\mu$  which covers the identity  $\text{id} : X \rightarrow X$  and fiberwise preserves the basepoint.  
(It is not required that  $u$  is a fiberwise homotopy equivalence.)

Let  $\underline{\mathbb{R}}^k$  denote the trivial vector bundle  $X \times \mathbb{R}^k \rightarrow X$ .



**Definition 1.4.** Fix  $n \in \mathbb{Z}$ . Let  $\xi_0, \xi_1$  be two  $G$ -vector bundles over  $X$ , and let  $k_0$  and  $k_1$  be two non-negative integers such that  $k_i + n \geq 0$  for  $i = 0, 1$ . Then two morphisms

$$u_i : S^{\xi_i \oplus \mathbb{R}^{k_i}} \rightarrow S^{\xi_i \oplus \mathbb{R}^{k_i+n}}$$

are called equivalent, if there are objects  $\mu_i$  in  $\text{SPHB}^G(X)$  for  $i = 0, 1$  and isomorphisms of  $G$ -vector bundles  $v : \mu_0 \oplus \xi_0 \cong \mu_1 \oplus \xi_1$  such that the following diagram in  $\text{SPHB}^G$  commutes up to homotopy

$$\begin{array}{ccc} S^{\mu_0 \oplus \mathbb{R}^{k_1}} \wedge_X S^{\xi_0 \oplus \mathbb{R}^{k_0}} & \xrightarrow{id \wedge_X u_0} & S^{\mu_0 \oplus \mathbb{R}^{k_1}} \wedge_X S^{\xi_0 \oplus \mathbb{R}^{k_0+n}} \\ \downarrow & & \downarrow \\ S^{\mu_0 \oplus \xi_0 \oplus \mathbb{R}^{k_0+k_1}} & & S^{\mu_0 \oplus \xi_0 \oplus \mathbb{R}^{k_0+k_1+n}} \\ \downarrow & & \downarrow \\ S^{\mu_1 \oplus \mathbb{R}^{k_0}} \wedge_X S^{\xi_1 \oplus \mathbb{R}^{k_1}} & \xrightarrow{id \wedge_X u_1} & S^{\mu_1 \oplus \mathbb{R}^{k_0}} \wedge_X S^{\xi_1 \oplus \mathbb{R}^{k_1+n}} \end{array}$$

**Definition 1.5.** For a proper  $G$ -CW-complex  $X$  define

$$\pi_G^n(X) = \{\text{equivalence classes of morphisms } u \text{ as above}\}$$

By introducing triviality conditions on a  $G$ -CW pair, (considering morphisms which are fibrewise constant with the value the point at infinity), equivariant cohomotopy groups are extended to an equivariant cohomology theory with multiplicative structure.

We introduce a Burnside ring for infinite groups, making out of Segal's remark, part 1 in Theorem 1.1, our definition for finite groups:

**Definition 1.6.** Let  $G$  be a Group with a finite model for the classifying space for proper actions  $\underline{E}(G)$ . The Burnside ring for  $G$  is the 0-th equivariant cohomotopy ring of the classifying space for proper actions. In symbols

$$A(G) = \pi_G^0(\underline{E}(G))$$

Denote by  $A^{\text{lim}}(G) = \lim_{H \in \mathcal{FIN}} A(H)$  the inverse limit of the Burnside rings of the finite subgroups of  $G$ . Notice that this agrees with the 0,0-entry of the  $E^2$ -term of the equivariant Atiyah-Hirzebruch spectral sequence. The following relations between the Burnside ring and the inverse-limit Burnside ring are easy consequences of the rational collapse of the Atiyah-Hirzebruch spectral sequence:

**Lemma 1.7.** *Let  $G$  be a discrete group admitting a finite model for the classifying space for proper actions.*

- (i) *The edge Homomorphism  $e : A(G) \rightarrow A^{\text{lim}}(G)$  has nilpotent kernel and cokernel. Its kernel is the nilradical.*
- (ii) *The edge homomorphism gives an isomorphism between the set of prime ideals in  $A(G)$  and  $A^{\text{lim}}(G)$  (in fact an homeomorphism in the Zariski topology), by assigning a prime ideal  $I \subset A^{\text{lim}}(H)$  its inverse image  $e^{-1}(I) \in A(G)$ .*
- (iii) *The rationalized Burnside ring  $\pi_G^0(\underline{E}(G)) \otimes \mathbb{Q}$  does not contain nilpotent elements.*

The following result was proved in [20], see also [6], Theorem 13 in page 58.

**Theorem 1.8** (Segal Conjecture for families of finite subgroups). *Let  $G$  be a discrete group and  $\mathcal{F}$  be a family of finite subgroups of  $G$  closed under conjugation and under subgroups. Fix a finite proper  $G$ -CW complex  $X$ , a finite dimensional proper*

$G$ -CW complex  $Z$  whose isotropy subgroups lie in  $\mathcal{F}$  and have bounded order. Let  $f : X \rightarrow Z$  be a  $G$ -map. Regard  $\pi_G^0(X)$  as a module over  $\pi_G^0(Z)$  and set

$$I = I_{\mathcal{F},Z} = \ker(\pi_G^0(Z) \xrightarrow{\text{res}_G^H \circ i^*} \prod_{H \in \mathcal{F}} \pi_H^0(Z^0))$$

then

$$\lambda_{X,\mathcal{F},f}^m : \{\pi_G^m(X) / I^n \cdot \pi_G^m(X)\} \rightarrow \{\pi_G^m(E_{\mathcal{F}} \times X^{n-1})\}$$

is an isomorphism of pro-groups. Also, the inverse system

$$\{\pi_G^m((E_{\mathcal{F}}(G) \times X)^n)\}_{n \geq 1}$$

satisfies the Mittag-leffler condition. In particular

$$\lim^1 \pi_G^m((E_{\mathcal{F}} \times X)^n) = 0$$

and  $\lambda_{X,\mathcal{F},f}$  induces an isomorphism

$$\pi_G^m(X)_{\mathbb{I}_G} \xrightarrow{\cong} \pi_G^m(E_{\mathcal{F}} \times X) \cong \lim_n \pi_G^m((E_{\mathcal{F}} \times X)^n)$$

**Proposition 1.9.** *Let  $G$  be a discrete group satisfying Conditions -1.2.*

*There exists a “ Generalized Bartsch element ”  $w \in \pi_G^0(\underline{E}G)$  for which the map  $\pi_G^0(\underline{E}G) \rightarrow H_{\mathbb{Z}(\text{Or}(G))}^0(\underline{E}G, \pi_{\bullet}^0(\{\bullet\})) = \lim_{K \in \text{Sub}(G)} \pi_K^0(\{\bullet\}) \xrightarrow{\psi_M} \pi_M^0(\{\bullet\})$  given by the composition of the edge homomorphism and the structural map for the inverse limit maps  $w$  to a power of the element constructed in 1.1 for any maximal subgroup  $M$ .*

*Proof.* Let  $X_{M_i} \in \pi_{M_i}^0(\{\bullet\})$  be the Bartsch element constructed in Theorem 1.1, part 4. Put  $x = \{x_{M_i}\} \in \lim_H \pi_H^0(\{\bullet\})$ . Choose an element  $w$  and a power  $k$  such that  $w$  is mapped to  $x^k$  under the edge homomorphism.  $\square$

**Lemma 1.10.** *If there exists a prime number  $p$  such that all maximal subgroups in  $\mathcal{MAX}$  are finite  $p$ -groups, then for any finite proper  $G$ -CW complex  $X$ , the groups  $\pi_G^n(X) \otimes \mathbb{Z}_{\hat{p}}$  and  $\pi_G^n(X)_{\mathbb{I}_{G,\mathcal{MAX}}}$  are isomorphic.*

*Proof.* This is an argument which goes back to W. Lück. See [20], [6], Proof of Theorem 13 in pages 60–62. The idea is to prove in the notation of Theorem 1.8 that for any proper finite  $G$ -CW complex  $X$  and any map  $X \rightarrow Z$ , the systems

$$\lambda_{X,\mathbb{I}_p,f}^m : \{\pi_G^m(X) / I_p^n \cdot \pi_G^m(X)\}$$

(Given by the  $p$ -adical ideal)

and

$$\lambda_{X,\mathcal{F},f}^m : \{\pi_G^m(X) / I_{\mathcal{MAX},Z}^n \cdot \pi_G^m(X)\}$$

(Given by completion at the family ideal)

are pro-isomorphic. Since both functors have Mayer-Vietoris sequences, both of the systems satisfy the Mittag-Leffler condition and in view of the 5-lemma for pro-modules, [3], section 2, an inductive argument can be used to reduce the problem to the situation of  $X = G/H$ , and where  $H$  is a finite  $p$ -group.

In this case, there exists a commutative diagram

$$\begin{array}{ccc} \pi_G^0(Z) & \xrightarrow{f^*} & \pi_G^m(G/H) \\ & \text{ind}_{H \rightarrow G}^{\cong} \downarrow & \\ A(H) & \xrightarrow{\cong} & \pi_H^0(\{*\}) \end{array}$$

Due to Frobenius reciprocity, for any finite group  $H$ ,  $|H| \mid I_H^n \subset I_H^{n+1}$ . On the other hand, conditions -1.2 restrict the cardinalities of finite groups to powers of  $p$ . Therefore there exists a map of inverse systems

$$\{A(H)/I_H^n\}_n \rightarrow \{A(H)/p^n\}_n$$

which is a pro-isomorphism because for finite  $p$ -groups  $I_H$ -adical and  $p$ -adical completions agree, Theorem 1.1, part 4. Therefore, it suffices to prove that the ideal  $I_{\mathcal{MAX},f} \subset \pi_H^0(\{\bullet\})$ , and the ideal  $I_p$  satisfy that the ideal  $I_{\mathcal{MAX}}/I_p$  is nilpotent.

Since  $\pi_H^0(\{\bullet\})$  is a noetherian ring, this holds if the nilradicals agree. Let  $P = \mathcal{P}_{K,p}$  be a prime ideal containing  $I_p$ . We need to show that  $P$  contains the image of the structure map for  $H$

$$\lim_H I_H \cong \prod_{M \in \mathcal{MAX}} I_M \rightarrow I_H$$

Let  $n_0$  be a sufficiently high power of  $p$  divided by the cardinality of all subgroups in  $\mathcal{MAX}$ .

Notice that the  $H$ -set  $S$  formed by the disjoint union of  $n_0/|H|$  copies of  $H$  is free. Choose an isomorphism  $S \xrightarrow{u} \{1, \dots, r\}$  and a group homomorphism  $H \xrightarrow{\rho_u} S_r$  into the set of automorphisms (permutations) of this set. This defines an action of  $H$  on  $S_r$ . Denote by  $S_r[\rho_u]_H$  the  $H$ -set obtained by this procedure and notice that this does not depend on the isomorphism to  $\{1, \dots, r\}$ . Let  $Syl_p(S_r)$  be the  $p$ -Sylow subgroup of  $S_r$ . We point out that the action of  $H$  on  $S_r$  defines an action on the homogeneous space  $S_r/Syl_p(S_r)$  by  $h \cdot \bar{\sigma} = h \cdot \sigma$ .

We denote by  $[S_r]_H := S_r[\rho_u]$  and  $[S_r/Syl_p(S_r)]_H := S_r/Syl_p(S_r)[\rho_u]$ . Put  $x_H = ([S_r] - |S_r| \mid H/H)_H$ , for every  $H \in \mathcal{MAX}$ .

The image under the structure map for the subgroup  $H$  is  $[S_r] - |S_r| \mid [H/H]$  and  $[S_r/Syl_p(S_r)] - |S_r/Syl_p(S_r)| \mid [H/H]$ . Both elements are in the prime ideal  $\mathcal{P}_{K,p}$ . Since  $\varphi_K : A(H) \rightarrow \mathbb{Z}$  sends both elements to  $p\mathbb{Z}$ , and  $\varphi_K([S_r] - |S_r|) = |S_r|_H^K - |S_r|$  for  $k \neq \{1\}$ , we conclude that  $K = \{1\}$  or  $p \neq 0$ . If  $K = \{1\}$ , then  $I_H = \mathcal{P}_{\{1\},0} \subset \mathcal{P}_{\{1\},p}$ . If  $K \neq \{1\}$ , then  $p \neq 0$  is a prime number and  $\varphi_K([S_r/Syl_p(S_r)] - |S_r/Syl_p(S_r)|) \in p\mathbb{Z}$ . Since  $|S_r/Syl_p(S_r)|$  is prime to  $p$ , and  $p$  divides the difference  $|S_r/Syl_p(S_r)|^K - |S_r/Syl_p(S_r)|$ , we conclude that  $K$  is a  $p$ -group and  $\mathcal{P}_{K,p} = \mathcal{P}_{\{1\},p}$ . Hence  $I_H = \mathcal{P}_{\{1\},0} \subset \mathcal{P}_{\{1\},p}$ .  $\square$

**Corollary 1.11.** *Let  $p$  be a prime number. For any group satisfying conditions -1.2 for which the maximal finite subgroups are finite  $p$ -groups, the groups  $\pi_G^0(\underline{EG}) \otimes \mathbb{Z}_{\hat{p}}$  and  $\pi_G^n(\underline{EG})_{\mathbb{I}_{G,\mathcal{MAX}}}$  are isomorphic.*

**Definition 1.12.** Let  $X$  be a  $G$ -CW complex of finite type. Define

$$\hat{\pi}_G^0(X) = \lim_n \pi_G^n(X_n) \otimes \mathbb{Q}_{\hat{p}}$$

From this point on, the proof of the proper mountain pass theorem follows quite close the arguments in [9], with the difference that we use equivariant cohomotopy with  $p$ -adic rational coefficients instead of equivariant cohomotopy with  $p$ -adic integral coefficients.

**Proposition 1.13.** *Let  $G$  be a discrete group satisfying -1.2. Let  $X$  be a proper  $G$ -CW complex of finite type which is contractible in the non equivariant sense. Then The map  $X \rightarrow \underline{EG}$  induces an isomorphism*

$$\hat{\pi}_G^0(\underline{EG}) \xrightarrow{\cong} \hat{\pi}_G^0(X)$$

*Proof.* The point is the existence of long exact sequences for the functor  $\hat{\pi}_G^*(X, A)$ , which is guaranteed by the natural equivalence with the Equivariant Cohomology Theory defined by  $(X, A) \mapsto \pi_G^m((E_{\mathcal{F}}, \phi) \times (X, A))$  on finite  $G$ -CW pairs.  $\square$

**Proposition 1.14.** *Let  $G$  be a group satisfying conditions -1.2. Let  $X$  be a proper  $G$ -CW complex which is nonequivariantly contractible. Then, there exists an element  $w \in \pi_G^0(\underline{EG}) \otimes \mathbb{Q}$  such that*

- $w \in \ker \pi_G^0(\underline{EG}) \otimes \mathbb{Q} \rightarrow \pi_G^0(G/H) \otimes \mathbb{Q}$  for all finite  $H$ .
- $w \in \ker \pi_G^0(\underline{EG}) \otimes \mathbb{Q} \rightarrow \pi_G^0(X_0) \otimes \mathbb{Q}$ .
- For every  $k > 0$  there exists an  $n > 0$  such that the image of  $w^k$  under  $\hat{\pi}_G^0(\underline{EG}) \rightarrow \pi_G^0(X_n) \otimes \mathbb{Q}_{\hat{p}}$  is not zero.

*Proof.* Let  $v \in \pi_G^0(\underline{EG}) \otimes \mathbb{Q} \cong \Pi_{H \in \mathcal{MAX}} A(H) \otimes \mathbb{Q}$  be the element constructed in proposition 1.1.

Let  $m = G\text{-cat}(X_0)$  and put  $w = v^m$ . As in [9], the following diagram commutes:

$$\begin{array}{ccc} \pi_G^0(\underline{EG}) \otimes \mathbb{Q}_{\hat{p}} & \xrightarrow{\quad\quad\quad} & \lim_n \pi_G^0(X_n) \otimes \mathbb{Q}_{\hat{p}} \\ \downarrow & & \downarrow \\ \pi_G^0(\underline{EG})_{\hat{\mathbb{I}}_{G, \mathcal{MAX}}} \otimes \mathbb{Q} & \xrightarrow{\cong} \hat{\pi}_G^0(\underline{EG}) \xrightarrow{\cong} & \hat{\pi}_G^0(X) \end{array}$$

as the left and right vertical maps are isomorphisms, and there are no nilpotent elements in the rationalized Burnside Ring  $\pi_G^0(\underline{EG}) \otimes \mathbb{Q}$ , there are no nilpotents in  $\pi_G^0(\underline{EG}) \otimes \mathbb{Q}_{\hat{p}}$ , and so there exists a natural number  $n$  such that the third condition holds.  $\square$

Let  $\hat{V} \subset E$  be a  $G$ -invariant linear subspace with a finite dimensional,  $G$ -invariant complement  $F_0$ . As a consequence of hypothesis 3 in Theorem -1.1, for any finite dimensional,  $G$ -invariant subspace  $\hat{F}$ , the sum  $F = F_0 \oplus \hat{F}$  satisfies

$$F - B_r(V) \subset \phi^a$$

**Lemma 1.15.** *There is a  $G$ -map  $f$  such that the diagram*

$$\begin{array}{ccc} (F, F - B_r(F)) & \xrightarrow{\quad\quad\quad} & (V - \{0\}, \phi^a) \\ \downarrow i_F & & \downarrow f \\ (F, F - S(F_0 \oplus F)) & \xrightarrow{j_F} & (V - \{0\}, S(\hat{V})) \end{array}$$

*Proof.* Compare lemma 5.2 in [13]. Define a map  $f : V \rightarrow \hat{V}$  by sending the bounded closed subspace  $A$  in theorem -1.1 to 0, mapping  $\hat{V} \cap \phi^a$  into  $\hat{V} - B_r(\hat{V})$  and extending to all of  $V$ , since  $\hat{V}$  is an absolute retract, Theorem 3.9 in page 1953 of [2].  $\square$

The same argument as in Proposition 5.3, [13], page 17 yields:

**Proposition 1.16.** *For any equivariant Cohomology Theory,  $\mathcal{H}_G^*$ ,*

$$G\text{-cat}(E, \phi^a) \geq \mathcal{H}_G^* \text{lenght}(S(F_0 \oplus F) \rightarrow S(F_0 \oplus \hat{E}, S(F_0)))$$

We now finish the proof of Theorem -1.1. This follows the proof of proposition 3.2 in [9].

**Proposition 1.17.**

$$G\text{-cat}(E, \phi^a) = \infty$$

*Proof.* Let  $F_n$  be an increasing sequence of finite dimensional linear  $G$ -subspaces of  $\hat{E}$  such that  $\hat{F} = \cup F_n$  is infinite dimensional. as in [9], the length of the inclusion

$$S(F_0 \oplus F_n) \rightarrow S(F_0 \oplus \hat{E}, S(F_0))$$

becomes arbitrarily large as  $n$  tends to infinity.

The (infinite)  $G$ -CW complex  $X = S(F_0 \oplus \hat{F})$  together with its filtration given by cellular decompositions of  $S(F_0 \oplus F_n)$  (of finite type) satisfy the hypotheses of proposition 1.13. Hence there is an element  $w \in \pi_G^0(\underline{E}G)$  satisfying conditions 1 to 3 in 1.13. let  $v$  and  $v_n$  be the images of  $W$  along the homomorphism induced by the universal maps  $S(F_0 \oplus \hat{E}) \rightarrow \underline{E}G$ , respectively  $S(F_0 \oplus \hat{F}_n) \rightarrow \underline{E}G$ . Since the diagram

$$\begin{array}{ccc} \pi_G^0(S(F_0 \oplus \hat{F}_n)) & \xrightarrow{j_n^*} & \pi_G^0(S(F_0 \oplus \hat{E}), S(F_0)) \\ & \nwarrow \quad \nearrow j^* & \\ & \pi_G^0(S(F_0 \oplus \hat{E})) & \end{array}$$

commutes up to homotopy,  $v_n \in \text{im}(j_n^*)$ , and proposition 1.13 yields that for any  $k$  there is an  $n$  with  $\hat{\pi}_G^* - \text{lenght} j_n \geq k$ . □

## 2. EXAMPLE

In this section, we shall discuss an example of Banach  $G$ -space with an action of an infinite group  $G$  satisfying the hypothesis in the Mountain Pass Theorem. We first illustrate an example of a problem lacking of the compactness assumptions on the critical set, typical in the case of non-compact symmetry.

**Example 2.1** (An improvement of a result by Rabinowitz). Let  $P$  be a  $p$ -group.  $\beta : P \rightarrow Gl_n(\mathbb{Z})$  be a group homomorphism. Consider The  $P$ -action on  $V := \mathbb{R}^n$  induced by  $\beta$ . Assume that

- The  $P$ -action on  $V := \mathbb{R}^n$  is free outside 0.
- There exist real numbers  $\{T_1, \dots, T_n\}$  (called in the subsequent periods) such that the abelian subgroup generated by the vectors  $(T_1, 0, \dots, 0), (0, T_2, 0, \dots, 0), \dots, T_i e_i, \dots, (0, 0, \dots, T_n)$  is  $P$ -invariant.

In this situation one has a proper action on  $V$  of the group  $\mathbb{Z}^n \rtimes P$  on the space  $\mathbb{R}^n$  furnished with the action induced by  $\beta$ .

Let  $T_0$  be a real number. Consider the Sobolev space  $E := W_{T_0}^{1,2}(\mathbb{R}, V)$  of  $T_0$ -periodic functions with the norm

$$|q| = \sqrt{\int_0^{T_0} |q|^2 dt + \left(\frac{1}{T_0} \int_0^{T_0} q(s) ds\right)^2}$$

Let  $L \in C^1(\mathbb{R} \times V, \mathbb{R}^{n^2})$ ,  $W \in C^1(\mathbb{R} \times V, \mathbb{R})$ ,  $f \in C(R, V)$  be such that

- For every  $t \in \mathbb{R}$  and  $q = (q_1, \dots, q_n)$   $L(t, q)$  is a symmetric matrix, and there exists an  $\alpha > 0$  such that  $\langle L(t, q)\xi, \xi \rangle \geq \alpha \langle \xi, \xi \rangle$ .
- $L$  is  $T_0$ -periodic in  $t$  and  $T_i$ -periodic in  $q_i$ .
- $W$  is  $T_0$ -periodic in  $t$  and  $T_i$ -periodic in  $q_i$ .
- $f$  is  $T_0$ -periodic in  $t$  and  $\frac{1}{T_0} \int_0^{T_0} f(t) dt = 0$ .

Consider the functional

$$\phi(q) := \int_0^{T_0} \left[ \frac{1}{2} L(t, q) \langle \dot{q}, \dot{q} \rangle - V(t, q) f(t) \right] dt$$

Notice that  $\phi$  is invariant under the proper action of the group  $G := \mathbb{Z}^n \rtimes P$ .

Let  $c$  be a real number and  $K_c := \{q \mid \phi'(q) = 0, \phi(q) = c\}$  be the critical set at level  $c$ . Let  $S$  be the subset consisting of those  $q = (q_1(t), \dots, q_n(t))$  for which  $0 < \frac{1}{T_0} \int_0^{T_0} q_i < T_i$ . Consider the copy of  $V$  given by  $\{u \in W_{T_0}^{1,2} \mid \frac{1}{T_0} \int_0^{T_0} u dt = u\}$  and denote by  $\hat{V}$  the orthogonal complement.

By an argument parallel to lemma 2.12 in [30], the set  $S := \{u = (u_i) \mid \forall i \in \{0, \dots, n\}, 0 \leq \frac{1}{T_0} \int_0^{T_0} u_i dt \leq T_i\}$  is a fundamental deformation region. The crucial point is the compactness of  $K_c \cap S$ . We reproduce the argument of Rabinowitz, [30], page 309.

There exists a description of  $\phi'$  as the functional  $D(P_1 + P)$ , where  $D : E \rightarrow E^*$  is the dualization map,  $P$  is a compact operator and  $\frac{d}{dt}P_1(u) = L(t, q)\dot{q} - \int_0^{T_0} L(\tau, q)\dot{q}d\tau$ .

Given a sequence  $Q_n = \xi_n + Y_n \subset K_c \cap S$ , with  $\xi_n \in V$  and  $Y_n \in \hat{V}$ , the ellipticity assumption and the definition of  $K_c \cap S$  imply that  $Y_n$  is bounded, and the definition of  $S$  imply that  $\xi_n$  is bounded. Hence, a subsequence  $Q_m$  converges weakly in  $E$  and strongly in  $L^\infty$  to  $Q$ ,  $P(Q_m)$  converges in  $E$ . The expression

$$\dot{Q}_n = L^{-1}(t, Q_m) \left[ \frac{1}{T_0} \int_0^{T_0} L(\tau, Q_m) \dot{Q}_m dt - \frac{d}{dt} P(Q_m) \right]$$

implies that the subsequence  $\dot{Q}_m$  converges in  $L^2$ , and  $\phi'(Q) = 0$ .

Moreover, the hypothesis on the action give that the critical set  $K_c$  is cocompact. This is due to the fact that  $G \backslash K_c$  is a quotient of the compact space  $P \backslash K_c \cap S$  after eventual identification. Theorem -1.1 Applied to the functional  $\phi$  and the subspace  $\hat{V}$ , the orthogonal complement of  $V$  in  $W_{T_0}^{1,2}(\mathbb{R}, \mathbb{R}^n)$  guarantee the existence of an unbounded sequence of critical values above a value  $a$  for which the mountain pass situation occurs. The critical points of  $\phi$  are classical solutions of the system of ordinary differential equations

$$\frac{d}{dt} L(t, q, \dot{q}) - \frac{1}{2} \frac{\partial L}{\partial q} \langle \dot{q}, \dot{q} \rangle + V_q(t, q) = f(t)$$

### 3. CONCLUDING REMARKS

Paraphrasing Willem, [34], page 3 Minimax-Type Theorems usually consist of different parts:

- Deformation lemma using some pseudogradient vector field.
- Construction of Palais-Smale typical sequences, which converge either due to some apriori compactness condition, or which give critical points using additional a posteriori information, typically *topological intersection properties*, like the intermediate value theorem, the Borsuk-Ulam theorem, degree notions, etc.

**Remark 3.1** (Borsuk-Ulam Type Theorems). In this work, the proof given by Bartsch-Clapp Puppe was adapted using an equivariant Borsuk-Ulam-Type Theorem, which may be deduced from 1.16 and 1.14. The problem of classifying the groups satisfying equivariant Borsuk-Ulam-Type theorem has deserved particular attention [8], [19], among others.

Let  $G$  be a discrete, linear group which acts properly and linearly on finite dimensional representation spheres  $S^V$ . Define the Borsuk-Ulam function  $b_G(n)$  as the maximal natural number  $k$  such that if there exists a  $G$ -map  $S^V \rightarrow S^W$  where  $\dim V \geq n$ , then  $\dim W \geq k$

**Problem 3.2.** *Classify all linear, discrete groups satisfying*

$$\lim_{n \rightarrow \infty} b_G(n) = \infty$$

as in [8], [19], and in this work, condition -1.2, the answer should involve restrictions for the number of primes dividing the cardinality of the isotropy groups.

**Remark 3.3** (Topological Noncompact Groups of Symmetry). In the context of Hamiltonian Systems, some proper actions of noncompact Lie groups appear [31]. The main contribution of [5] is the development of Equivariant Cohomotopy Theory for these class of symmetries. The use of Equivariant Algebraic Topology, particularly Equivariant Cohomotopy may be useful. However, in this context, the Segal Conjecture (which was the main homotopy theoretical input of theorem -1.1, crucially in the proof of the Borsuk-Ulam-type result) is not true, as it is not even true for compact Lie groups, see [16], [10].

**Remark 3.4** (Equivariant Degree Notions for Infinite Discrete Groups). In [6], [5], an equivariant degree notion for proper actions of discrete group is defined. This assigns to a quadruple  $(E, F, T, c)$  consisting of locally trivial  $G$ - Hilbert bundles over a proper, cocompact  $G$ -CW complex, a fibrewise Fredholm operator  $T$  and a fibrewise compact nonlinearity satisfying the property that the map  $T_x + c_x : E_x \rightarrow F_x$  defined on the fibers  $E_x, F_x$  over each point  $x$  is proper, an element in the equivariant cohomotopy  $\pi_G^*(X)$ , as defined in definition 1.3. We will analyze the applicability of this degree notion to equivariant variational problems elsewhere.

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